Binary Matrix Operations

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Transforming Numerical Methods Education for STEM Undergraduates
Binary Matrix Operations
Objectives

1. *add, subtract, and multiply matrices, and*

2. *apply rules of binary operations on matrices.*
Matrix Addition

Two matrices \(A\) and \(B\) can be added only if they are the same size. The addition is then shown as

\[
[C] = [A] + [B]
\]

where

\[
c_{ij} = a_{ij} + b_{ij}
\]
Add the following two matrices.

\[
[A] = \begin{bmatrix}
5 & 2 & 3 \\
1 & 2 & 7 \\
\end{bmatrix} \quad [B] = \begin{bmatrix}
6 & 7 & -2 \\
3 & 5 & 19 \\
\end{bmatrix}
\]
Example 1 (cont.)

\[
[C] = [A] + [B]
\]

\[
= \begin{bmatrix}
5 & 2 & 3 \\
1 & 2 & 7
\end{bmatrix}
+ \begin{bmatrix}
6 & 7 & -2 \\
3 & 5 & 19
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5+6 & 2+7 & 3-2 \\
1+3 & 2+5 & 7+19
\end{bmatrix}
\]

\[
= \begin{bmatrix}
11 & 9 & 1 \\
4 & 7 & 26
\end{bmatrix}
\]
Example 2

Blowout r’us store has two store locations and, and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

\[
[A] = \begin{bmatrix}
25 & 20 & 3 & 2 \\
5 & 10 & 15 & 25 \\
6 & 16 & 7 & 27 \\
\end{bmatrix} \quad [B] = \begin{bmatrix}
20 & 5 & 4 & 0 \\
3 & 6 & 15 & 21 \\
4 & 1 & 7 & 20 \\
\end{bmatrix}
\]

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?
Example 2 (cont.)

\[
[C] = [A] + [B]
\]

\[
\begin{bmatrix}
25 & 20 & 3 & 2 \\
5 & 10 & 15 & 25 \\
6 & 16 & 7 & 27
\end{bmatrix} + \begin{bmatrix}
20 & 5 & 4 & 0 \\
3 & 6 & 15 & 21 \\
4 & 1 & 7 & 20
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(25+20) & (20+5) & (3+4) & (2+0) \\
(5+3) & (10+6) & (15+15) & (25+21) \\
(6+4) & (16+1) & (7+7) & (27+20)
\end{bmatrix}
\]
The answer then is,

\[
\begin{bmatrix}
45 & 25 & 7 & 2 \\
8 & 16 & 30 & 46 \\
10 & 17 & 14 & 47 \\
\end{bmatrix}
\]

So if one wants to know the total number of Copper tires sold in quarter 4 at the two locations, we would look at Row 3 – Column 4 to give \( c_{34} = 47 \).
Matrix Subtraction

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size. The subtraction is then given by

$$[D] = [A] - [B]$$

Where

$$d_{ij} = a_{ij} - b_{ij}$$
Example 3

Subtract matrix \([B]\) from matrix \([A]\).

\[
[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}
\]

\[
[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}
\]
Example 3 (cont.)

\[
[D] = [A] - [B]
\]

\[
= \begin{bmatrix}
5 & 2 & 3 \\
1 & 2 & 7
\end{bmatrix} - \begin{bmatrix}
6 & 7 & -2 \\
3 & 5 & 19
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(5 - 6) & (2 - 7) & (3 - (-2)) \\
(1 - 3) & (2 - 5) & (7 - 19)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 & -5 & 5 \\
-2 & -3 & -12
\end{bmatrix}
\]
Example 4

Blowout r’us store has two store locations and , and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

\[
\begin{bmatrix}
25 & 20 & 3 & 2 \\
5 & 10 & 15 & 25 \\
6 & 16 & 7 & 27 \\
\end{bmatrix}
\quad
\begin{bmatrix}
20 & 5 & 4 & 0 \\
3 & 6 & 15 & 21 \\
4 & 1 & 7 & 20 \\
\end{bmatrix}
\]

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4. What are the total tire sales for the two locations by make and quarter?
Example 4 (cont.)

\[
[D] = [A] - [B]
\]

\[
= \begin{bmatrix}
25 & 20 & 3 & 2 \\
5 & 10 & 15 & 25 \\
6 & 16 & 7 & 27
\end{bmatrix}
+ \begin{bmatrix}
20 & 5 & 4 & 0 \\
3 & 6 & 15 & 21 \\
4 & 1 & 7 & 20
\end{bmatrix}
= \begin{bmatrix}
25 - 20 & 20 - 5 & 3 - 4 & 2 - 0 \\
5 - 3 & 10 - 6 & 15 - 15 & 25 - 21 \\
6 - 4 & 16 - 1 & 7 - 7 & 27 - 20
\end{bmatrix}
\]
Example 4 (cont.)

The answer then is,

\[
\begin{bmatrix}
5 & 15 & -1 & 2 \\
2 & 4 & 0 & 4 \\
2 & 15 & 0 & 7 \\
\end{bmatrix}
\]

So if you want to know how many more Copper tires were sold in quarter 4 in store A than store B, \(d_{34} = 7\). Note that \(d_{13} = -1\) implies that store A sold 1 less in Michigan tire than store B in quarter 3.
Two matrices \([A]\) and \([B]\) can be multiplied only if the number of columns of \([A]\) is equal to the number of rows of \([B]\) to give

\[
[C]_{m \times n} = [A]_{m \times p} [B]_{p \times n}
\]

If \([A]\) is a \(m \times p\) matrix and \([B]\) is a \(p \times n\) matrix, the resulting matrix \([C]\) is a \(m \times n\) matrix.
So how does one calculate the elements of \([C]\) matrix?

\[
c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}
\]

\[
= a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ip} b_{pj}
\]

for each \(i = 1, 2, \ldots, m\) and \(j = 1, 2, \ldots, n\)

To put it in simpler terms, the \(i^{th}\) row and \(j^{th}\) column of the \([C]\) matrix in \([C] = [A][B]\) is calculated by multiplying the \(i^{th}\) row of \([A]\) by the \(j^{th}\) column of \([B]\).
Example 5

Given the following two matrices,

\[ [A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} \quad \quad [B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix} \]

Find their product,

\[ [C] = [A][B] \]
Example 5 (cont.)

$c_{12}$ be found by multiplying the first row of $[A]$ by the second column of $[B],$

$$c_{12} = \begin{bmatrix} 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$

$$= -56$$
Similarly, one can find the other elements of \( [C] \) to give

\[
[C] = \begin{bmatrix}
52 & -56 \\
76 & -88
\end{bmatrix}
\]
Example 6

Blowout r’us store has two store locations and, and their sales of tires are given by make (in rows) and quarters (in columns) as shown below.

\[
\begin{bmatrix}
25 & 20 & 3 & 2 \\
5 & 10 & 15 & 25 \\
6 & 16 & 7 & 27 \\
\end{bmatrix}
\begin{bmatrix}
20 & 5 & 4 & 0 \\
3 & 6 & 15 & 21 \\
4 & 1 & 7 & 20 \\
\end{bmatrix}
\]

where the rows represent the sale of Tirestone, Michigan and Copper tires respectively and the columns represent the quarter number: 1, 2, 3 and 4.
Find the per quarter sales of store $A$ if the following are the prices of each tire.

Tirestone = $33.25  
Michigan = $40.19  
Copper = $25.03

The answer is given by multiplying the price matrix by the quantity of sales of store $A$. The price matrix is $[33.25 \quad 40.19 \quad 25.03]$. 
Therefore, the per quarter sales of store $A$ dollars is given by the four columns of the row vector

$$[C] = \begin{bmatrix} 1182.38 & 1467.38 & 877.81 & 1747.06 \end{bmatrix}$$

Remember since we are multiplying a $1 \times 3$ matrix by a $3 \times 4$ matrix, the resulting matrix is a $1 \times 4$ matrix.
If $[A]$ is a $n \times n$ matrix and $k$ is a real number, then the scalar product of $k$ and $[A]$ is another $n \times n$ matrix $[B]$, where $b_{ij} = k a_{ij}$. 
Example 7

Given the matrix,

\[
[A] = \begin{bmatrix}
2.1 & 3 & 2 \\
5 & 1 & 6
\end{bmatrix}
\]

Find \(2[A]\)
The solution to the product of a scalar and a matrix by the following method,

\[
2[A] = 2 \begin{bmatrix}
2.1 & 3 & 2 \\
5 & 1 & 6 \\
\end{bmatrix}
= \begin{bmatrix}
2 \times 2.1 & 2 \times 3 & 2 \times 2 \\
2 \times 5 & 2 \times 1 & 2 \times 6 \\
\end{bmatrix}
= \begin{bmatrix}
4.2 & 6 & 4 \\
10 & 2 & 12 \\
\end{bmatrix}
\]
If $[A_1], [A_2], \ldots, [A_p]$ are matrices of the same size and $k_1, k_2, \ldots, k_p$ are scalars, then $k_1[A_1] + k_2[A_2] + \ldots + k_p[A_p]$ is called a linear combination of $[A_1], [A_2], \ldots, [A_p]$. 
Example 8

If

\[
[A_1] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix}, [A_2] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}, [A_3] = \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}
\]

then find

\[
[A_1] + 2[A_2] - 0.5[A_3]
\]
Example 8 (cont.)

\[
[A_1] + 2[A_2] - 0.5[A_3]
\]

\[
= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix} - 0.5 \begin{bmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{bmatrix}
\]

\[
= \begin{bmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 1.1 & 1 \\ 1.5 & 1.75 & 3 \end{bmatrix}
\]

\[
= \begin{bmatrix} 9.2 & 10.9 & 5 \\ 11.5 & 2.25 & 10 \end{bmatrix}
\]
**Binary Matrix Operations**

**Commutative law of addition**

If \([A]\) and \([B]\) are \(m \times n\) matrices, then

\[
[A] + [B] = [B] + [A]
\]

**Associative law of addition**

If \([A]\), \([B]\) and \([C]\) are \(m \times n\), \(n \times p\), and \(p \times r\) size matrices, respectively, then

\[
[A] + ([B] + [C]) = ([A] + [B]) + [C]
\]

**Associative law of multiplication**

If \([A]\), \([B]\) and \([C]\) are all \(m \times n\), \(n \times p\) and \(p \times r\) size matrices, respectively, then

\[
[A]([B][C]) = ([A][B])[C]
\]

and the resulting matrix size on both sides of the equation is \(m \times r\).
Distributive Law

If $A$ and $B$ are $m \times n$ matrices, and $C$ and $D$ are $n \times p$ size matrices

$$A([C]+[D])=A[C]+A[D]$$


and the resulting matrix size on both sides of the equation is $m \times r$. 
Example 9

Illustrate the associative law of multiplication of matrices using

\[
[A] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}, \quad [B] = \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}, \quad [C] = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}
\]
Example 9 (cont.)

\[
[B][C] = \\
= \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \\
= \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix}
\]

\[
[A]([B][C]) = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix} \\
= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}
\]
Example 9 (cont.)

\[
[A][B] = \begin{bmatrix}
1 & 2 \\
3 & 5 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 5 \\
9 & 6
\end{bmatrix}
= \begin{bmatrix}
20 & 17 \\
51 & 45 \\
18 & 12
\end{bmatrix}
\]

\[
([A][B])[C] = \begin{bmatrix}
20 & 17 \\
51 & 45 \\
18 & 12
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
3 & 5
\end{bmatrix}
= \begin{bmatrix}
91 & 105 \\
237 & 276 \\
72 & 78
\end{bmatrix}
\]
Is \([A][B]=[B][A]\)?

If \([A][B]\) exists, number of columns of \([A]\) has to be same as the number of rows of \([B]\) and if \([B][A]\) exists, number of columns of \([B]\) has to be same as the number of rows of \([A]\).

Now for \([A][B]=[B][A]\), the resulting matrix from \([A][B]\) and \([B][A]\) has to be of the same size. This is only possible if \([A]\) and \([B]\) are square and are of the same size. Even then in general \([A][B] \neq [B][A]\).
Example 10

Determine if

\([A][B]=[B][A]\)

for the following matrices

\[
[A] = \begin{bmatrix}
6 & 3 \\
2 & 5
\end{bmatrix}, \quad [B] = \begin{bmatrix}
-3 & 2 \\
1 & 5
\end{bmatrix}
\]
Example 10 (cont.)

\[
[A][B] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix}
\]

\[
[B][A] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}
\]

Therefore

\[
[A][B] \neq [B][A]
\]