Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Chapter 4.05
System of Equations
COMPLETE SOLUTION SET

1. A 3×4 matrix can have a rank of at most
   (A) 3
   (B) 4
   (C) 5
   (D) 12

Solution
The correct answer is (*A*).

Since the largest square submatrix of a 3×4 matrix can be 3, a 3×4 matrix can have a rank of at most 3.

2. Three kids – Jim, Corey and David receive an inheritance of \$2,253,453. The money is put in three trusts but is not divided equally to begin with. Corey gets three times what David gets because Corey made an "A" in Dr. Kaw's class. Each trust is put in an interest generating investment. The three trusts of Jim, Corey and David pay an interest of 6%, 8%, 11%, respectively. The total interest of all the three trusts combined at the end of the first year is \$190,740.57. How much money was invested in each trust? The equations in a matrix form are

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ .06 & .08 & .11 \end{bmatrix} \begin{bmatrix} J \\ C \\ D \end{bmatrix} = \begin{bmatrix} 2253543 \\ 0 \\ 190740.57 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ .06 & .08 & .11 \end{bmatrix} \begin{bmatrix} J \\ C \\ D \end{bmatrix} = \begin{bmatrix} 2253543 \\ 0 \\ 190740.57 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 6 & 8 & 11 \end{bmatrix} \begin{bmatrix} J \\ C \\ D \end{bmatrix} = \begin{bmatrix} 2253543 \\ 0 \\ 190740.57 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ .06 & .08 & .11 \end{bmatrix} \begin{bmatrix} J \\ C \\ D \end{bmatrix} = \begin{bmatrix} 2253543 \\ 0 \\ 190740.57 \end{bmatrix}$

Solution

The correct answer is (*B*).

Let J, C, and D be the inheritance portions of Jim, Corey and David, respectively.

The total inheritance is \$2,253,453 gives

$$J + C + D = 2{,}253{,}453 \qquad (1)$$

Corey's trust is three times that of David's

$$C = 3D$$

gives

$$C - 3D = 0 \qquad (2)$$

The three trusts of Jim, Corey and David pay an interest of 6%, 8%, and 11%, respectively. The total interest of all the three trusts combined at the end of the first year is $190,740.57.

The total interest is

$$\frac{6}{100}J+\frac{8}{100}C+\frac{11}{100}D=190{,}740.57$$

and gives

$$0.06J+0.08C+0.11D=190{,}740.57 \qquad (3)$$

Equations (1) - (3) can be rewritten as

$$\begin{aligned}
&1J+1C+1D=2{,}253{,}453\\
&0J+1C-3D=0\\
&0.06J+0.08C+0.11D=190{,}740.57
\end{aligned}$$

Setting the three equations in matrix form gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0.06 & 0.08 & 0.11 \end{bmatrix}\begin{bmatrix} J \\ C \\ D \end{bmatrix}=\begin{bmatrix} 2{,}253{,}453 \\ 0 \\ 190{,}740.57 \end{bmatrix}$$

3. Which of the following matrices does not have an inverse

(A) $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

(B) $\begin{bmatrix} 6 & 7 \\ 12 & 14 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 6 \\ 7 & 0 \end{bmatrix}$

Solution

The correct answer is (B).

The matrix in choice (A) is

$$[A] = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 5\times 8 - 6\times 7 \\ &= 40 - 42 \\ &= -2 \\ &\neq 0 \end{aligned}$$

So matrix $[A]$ has an inverse.

The matrix in choice (B) is

$$[B] = \begin{bmatrix} 6 & 7 \\ 12 & 14 \end{bmatrix}$$

$$\begin{aligned} \det(B) &= 6\times 14 - 7\times 12 \\ &= 84 - 84 \\ &= 0 \end{aligned}$$

So matrix $[B]$ does not have an inverse.

The matrix in choice C is

$$[C] = \begin{bmatrix} 6 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\det(C) = 6\times 7 - 0\times 0$$

$$= 42 - 0$$
$$= 42$$
$$\neq 0$$

So matrix $[C]$ has an inverse.

The matrix in choice D is

$$[D] = \begin{bmatrix} 0 & 6 \\ 7 & 0 \end{bmatrix}$$

$$\begin{aligned} \det(D) &= 0 \times 0 - 6 \times 7 \\ &= 0 - 42 \\ &= -42 \\ &\neq 0 \end{aligned}$$

So matrix $[D]$ has an inverse.

4. The set of equations

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 8 & 19 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 26 \\ 70 \end{bmatrix}$$

has

(A) no solution
(B) finite number of solutions
(C) a unique solution
(D) infinite solutions

Solution

The correct answer is (*D*).

First, find the rank of the coefficient matrix

$$[A] = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 7 \\ 5 & 8 & 19 \end{bmatrix}$$

The largest square submatrix possible is of order 3 and is $[A]$ itself. However, the $\det(A) = 0$. One of the next largest square submatrix possible is of order 2.

$$\det\left(\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}\right) = 1 \neq 0$$

Thus at least one submatix has a determinant that is not equal to zero. Hence, $\text{rank}(A) = 2$.

Next find the rank of the augmented matrix.

$$[B] = \begin{bmatrix} 1 & 2 & 5 & \vdots & 8 \\ 2 & 3 & 7 & \vdots & 26 \\ 5 & 8 & 19 & \vdots & 70 \end{bmatrix}$$

Since there are no square submatrices of order 4×4 as $[B]$ is a 3×4 matrix, the rank of $[B]$ is at most 3. So let us look at the square submatrices of $[B]$ of order 3×3; if any of these square submatrices have a determinant not equal to zero, then the rank is 3. However,

$$\det\left(\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 7 \\ 8 & 19 & 70 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2 & 5 & 18 \\ 3 & 7 & 26 \\ 8 & 19 & 70 \end{bmatrix}\right) = 0$$

The next largest square submatrix possible is of order 2×2. One of these 2×2 submatrix of $[B]$ is

$$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

and

$$\det\left(\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}\right) = 1 \neq 0$$

Thus the rank of $[B] = 2$. $\text{Rank}(A) = \text{rank}(B)$ implies that the system of equations is consistent and has at least one solution. But $\text{Rank}(A) <$ number of unknowns, which implies that the set of equations has infinite number of solutions.

5. Given a system of $[A][X]=[C]$ where $[A]$ is $n\times n$ matrix and $[X]$ and $[C]$ are $n\times 1$ matrices, a unique solution $[X]$ exists if
   (A) rank of $[A]=$ rank of $[A\vdots C]$
   (B) rank of $[A]=$ rank of $[A\vdots C]= n$
   (C) rank of $[A]<$ rank of $[A\vdots C]$
   (D) rank of $[A]=$ rank of $[A\vdots C]<n$

Solution

The correct answer is (*B*).

For a system of equations $[A]_{n\times n}[X]_{n\times 1}=[C]_{n\times 1}$, the rank of $[A]$ has to be equal to the rank of $[A\vdots C]$ for the system of equations to be consistent. Then for the solution to be unique, this rank has to be equal to the number of unknowns.

6. If $[A][X]=\begin{bmatrix}-13\\76\\38\end{bmatrix}$ and

$$[A]^{-1}=\begin{bmatrix}1 & 2 & -4\\-8 & 2 & 16\\2 & 4 & 8\end{bmatrix}$$

then

(A) $[X]=\begin{bmatrix}-13.000\\864.00\\582.00\end{bmatrix}$

(B) one cannot find a unique $[X]$.

(C) $[X]=\begin{bmatrix}-1.0000\\2.0000\\4.0000\end{bmatrix}$

(D) no solutions of $[X]$ are possible

Solution

The correct answer is (A).

Since $[A]^{-1}$ exists, $[A][X]=[C]$ has a unique solution.

Given

$$[A]^{-1}=\begin{bmatrix}1 & 2 & -4\\-8 & 2 & 16\\2 & 4 & 8\end{bmatrix}$$

we can find

$$[A]=\begin{bmatrix}-0.16667 & -0.11111 & .013889\\0.33333 & 0.055556 & 0.055556\\-0.12500 & 0.00000 & 0.062500\end{bmatrix}$$

Solving $[A][X]=[C]$, we get

$$[X]=\begin{bmatrix}-13.000\\864.00\\582.00\end{bmatrix}$$

One does not need to do this as

$$[A][X]=[C]$$

$$[X]=[A]^{-1}[C]$$

$$=\begin{bmatrix} 1 & 2 & -4 \\ -8 & 2 & 16 \\ 2 & 4 & 8 \end{bmatrix}\begin{bmatrix} -13 \\ 76 \\ 38 \end{bmatrix}$$

$$=\begin{bmatrix} -13.000 \\ 864.00 \\ 582.00 \end{bmatrix}$$