Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
LU Decomposition Method
Simultaneous Linear Equations
COMPLETE SOLUTION SET

1. The $[L][U]$ decomposition method is computationally more efficient than Naïve Gauss elimination for solving
> (A) a single set of simultaneous linear equations.
> (B) multiple sets of simultaneous linear equations with different coefficient matrices and the same right hand side vectors.
> (C) multiple sets of simultaneous linear equations with the same coefficient matrix and different right hand side vectors.
> (D) less than ten simultaneous linear equations.

Solution
The correct answer is (C).

An example can be demonstrated by finding the inverse of the matrix $[A]$. The problem of finding the inverse reduces to solving *n* sets of equations with the coefficient matrix staying the same and *n* different RHS vectors. These *n* RHS vectors are the *n* columns of the identity matrix.

For calculations of each column of the inverse of the $[A]$ matrix, the coefficient matrix $[A]$ in the set of equations $[A][X]=[C]$ does not change. So if we use the LU Decomposition method, the $[A]=[L][U]$ decomposition needs to be done only once and forward substitution $[L][Z]=[C]$ and the back substitution $[U][X]=[Z]$ still needs to be done *n* times.

The total computational time $CT\mid_{inverse\,LU}$ required to find the inverse of a matrix using LU decomposition is

$$CT\mid_{inverse\,LU} = 1 \times CT\mid_{DE} + n \times CT\mid_{FS} + n \times CT\mid_{BS}$$

$$= 1 \times T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right) + n \times T\left(4n^2 - 4n\right) + n \times T\left(4n^2 + 12n\right)$$

$$= T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right)$$

where
the computational time[1] $CT\mid_{DE}$ to decompose the $[A]$ matrix to $[L][U]$ form is given by

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[1] The time is calculated by first separately calculating the number of additions, subtractions, multiplications, and divisions in a procedure such as back substitution, etc. We then assume 4

$$CT\big|_{DE} = T\left(\frac{8n^3}{3} + 4n^2 - \frac{20n}{3}\right),$$

the computational time $CT\big|_{FS}$ to solve $[L][Z]=[C]$ by forward substitution is given by

$$CT\big|_{FS} = T\left(4n^2 - 4n\right),\text{ and}$$

the computational time $CT\big|_{BS}$ to solve $[U][X]=[Z]$ by back substitution is given by

$$CT\big|_{BS} = T\left(4n^2 + 12n\right)$$

In comparison, if Gaussian elimination method were used to find the inverse of matrix $[A]$, the forward elimination as well as the back substitution will have to be done n times. The total computational time $CT\big|_{inverse\,GE}$ required to find the inverse of a matrix by using Gaussian elimination then is

$$CT\big|_{inverse\,GE} = n \times CT\big|_{FE} + n \times CT\big|_{BS}$$

$$= n \times T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right) + n \times T\left(4n^2 + 12n\right)$$

$$= T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right)$$

where
the computational time $CT\big|_{FE}$ for the forward elimination part is given by

$$CT\big|_{FE} = T\left(\frac{8n^3}{3} + 8n^2 - \frac{32n}{3}\right),\text{ and}$$

the computational time $CT\big|_{BS}$ for the back substitution part is given by

$$CT\big|_{BS} = T\left(4n^2 + 12n\right)$$

Clearly for large n, $CT\big|_{inverse\,GE} \gg CT\big|_{inverse\,LU}$ as $CT\big|_{inverse\,GE}$ has the dominating terms of $8n^4/3$ and $CT\big|_{inverse\,LU}$ has the dominating terms of $32n^3/3$. For large values of n, Gaussian elimination method would take more computational time (approximately $n/4$ times – prove it) than the LU decomposition method. Typical values of the ratio of the computational time for different values of n are given in Table 1.

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$CT\big|_{inverse\,GE}/CT\big|_{inverse\,LU}$	3.28	25.83	250.8	2501

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clock cycles each for an add, subtract, or multiply operation, and 16 clock cycles for a divide operation as is the case for a typical AMD®-K7 chip.
http://www.isi.edu/~draper/papers/mwscas07_kwon.pdf

2. The lower triangular matrix $[L]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 8 & 12 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.5000 & 1 \end{bmatrix}$

Solution
The correct answer is (A).

We must first complete the first step of forward elimination.

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix}$$

First step: Multiply *Row 1* by $\dfrac{10}{25} = 0.4$, and subtract the results from *Row 2*

$$[Row\,2] - [Row\,1] \times (0.4) = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 8 & 12 & 22 \end{bmatrix}$$

Multiply *Row 1* by $\dfrac{8}{25} = 0.32$, and subtract the results from *Row 3*

$$[Row\,3]-[Row\,1]\times(0.32) = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.4 \\ 0 & 10.4 & 20.72 \end{bmatrix}$$

To find ℓ_{21} and ℓ_{31}, what multiplier was used to make the a_{21} and a_{31} elements zero in the first step of forward elimination using the Naïve Gauss elimination method? They are

$$\ell_{21} = 0.4$$

$$\ell_{31} = 0.32$$

To find ℓ_{32}, what multiplier would be used to make the a_{32} element zero? Remember the a_{32} element is made zero in the second step of forward elimination.

So

$$\ell_{32} = \frac{10.4}{6} = 1.7333$$

Hence

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$$

3. The upper triangular matrix $[U]$ in the $[L][U]$ decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(A)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$$

(B)
$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$$

(C)
$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$$

(D)
$$\begin{bmatrix} 1 & 0.2000 & 0.16000 \\ 0 & 1 & 2.4000 \\ 0 & 0 & -4.240 \end{bmatrix}$$

Solution
The correct answer is (C).

The $[U]$ matrix is the same as the coefficient matrix that is found at the end of the forward elimination steps of the Naïve Gauss elimination method, that is

$$[U] = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 12 & 22 \end{bmatrix}$$

<u>First step</u>: The first step of forward elimination does not need to be conducted as a_{21} and a_{31} are already zero.

<u>Second step</u>: Multiply *Row 2* by $\dfrac{12}{8} = 1.5$ and subtract the results from *Row 3*

$$[Row\,3] - [Row\,2] \times (1.5) = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$$

Thus,

$$[U] = \begin{bmatrix} 25 & 5 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & -2 \end{bmatrix}$$

4. For a given 2000×2000 matrix $[A]$, assume that it takes about 15 seconds to find the inverse of $[A]$ by the use of the $[L][U]$ decomposition method, that is, finding the $[L][U]$ once, and then doing forward substitution and back substitution 2000 times using the 2000 columns of the identity matrix as the right hand side vector. The approximate time, in seconds, that it will take to find the inverse if found by repeated use of the Naive Gauss elimination method, that is, doing forward elimination and back substitution 2000 times by using the 2000 columns of the identity matrix as the right hand side vector is most nearly

(A) 300
(B) 1500
(C) 7500
(D) 30000

Solution
The correct solution is (C).

From the solution of Problem 1 of this MC question test, the computational times for finding the inverse of $[A]$ by Naïve Gaussian elimination and LU decomposition is given by

$$T\left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3}\right) \text{ and } T\left(\frac{32n^3}{3} + 12n^2 - \frac{20n}{3}\right), \text{ respectively, where}$$

T = clock cycle time
n = number of rows or columns in square matrix $[A]$.

The ratio of computational time, R between the two methods then is

$$R \approx \frac{T\left(\dfrac{8n^4}{3} + 12n^3 + \dfrac{4n^2}{3}\right)}{T\left(\dfrac{32n^3}{3} + 12n^2 - \dfrac{20n}{3}\right)}$$

For a 2000×2000 matrix

$$R \approx \frac{T\left(\dfrac{8 \times 2000^4}{3} + 12 \times 2000^3 + \dfrac{4 \times 2000^2}{3}\right)}{T\left(\dfrac{32 \times 2000^3}{3} + 12 \times 2000^2 - \dfrac{20 \times 2000}{3}\right)}$$

$$= 500.84$$

Since it takes 15 seconds to find the inverse by LU decomposition, it would take about $500.84 \times 15 = 7512.6$ seconds to find the inverse by using Naïve Gaussian elimination.

5. The algorithm for solving a set of n equations $[A][X]=[C]$, where $[A]=[L][U]$ involves solving $[L][Z]=[C]$ by forward substitution. The algorithm to solve $[L][Z]=[C]$ is given by

(A) $z_1 = c_1/l_{11}$
   for i from 2 to n do
   sum $= 0$
   for j from 1 to i do
   sum $=$ sum $+ l_{ij} * z_j$
   end do
   $z_i = (c_i - sum)/l_{ii}$
   end do

(B) $z_1 = c_1/l_{11}$
   for i from 2 to n do
   sum $= 0$
   for j from 1 to $(i-1)$ do
   sum $=$ sum $+ l_{ij} * z_j$
   end do
   $z_i = (c_i - sum)/l_{ii}$
   end do

(C) $z_1 = c_1/l_{11}$
   for i from 2 to n do
   for j from 1 to $(i-1)$ do
   sum $=$ sum $+ l_{ij} * z_j$
   end do
   $z_i = (c_i - sum)/l_{ii}$
   end do

(D) for i from 2 to n do
   sum $= 0$
   for j from 1 to $(i-1)$ do
   sum $=$ sum $+ l_{ij} * z_j$
   end do
   $z_i = (c_i - sum)/l_{ii}$
   end do

Solution
The correct answer is (B).
Since $[L][Z]=[C]$ is written as

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\ell_{21} & 1 & & \vdots \\
\ell_{31} & & \ddots & 0 \\
\vdots & & & \\
\ell_{n1} & \ell_{n2} & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n
\end{bmatrix}
=
\begin{bmatrix}
c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n
\end{bmatrix}
$$

We know that the first step of forward substitution is

$$z_1 = c_1 / l_{11}$$

The next equations would be

$$z_2 = \frac{c_2 - \ell_{21} \times z_1}{\ell_{22}}$$

$$z_3 = \frac{c_3 - (\ell_{31} \times z_1 + \ell_{32} \times z_2)}{\ell_{33}}$$

$$\vdots$$

$$z_i = \frac{c_i - \sum_{j=1}^{i-1}(\ell_{ij} \times z_j)}{\ell_{ii}}$$

Thus, the algorithm is

$z_1 = c_1 / l_{11}$
for i from 2 to n do
   sum = 0   (the variable *sum* must be cleared each time through the loop)

   for j from 1 to (i-1) do   (this loop finds $\sum_{j=1}^{i-1}(\ell_{ij} \times z_j)$)

      sum = sum $+ l_{ij} * z_j$

   end do
 $z_i = (c_i - \text{sum}) / l_{ii}$
end do

6. To solve boundary value problems, a numerical method based on finite difference method is used. This results in simultaneous linear equations with tridiagonal coefficient matrices. These are solved using a specialized $[L][U]$ decomposition method.

Choose the set of equations that approximately solves the boundary value problem

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2, \ y(0) = 0, \ y(12) = 0, \ 0 \le x \le 12$$

The second derivative in the above equation is approximated by the second-order accurate central divided difference approximation as learned in the differentiation module (Chapter 02.02). A step size of $h = 4$ is used, and hence the value of y can be found approximately at equidistantly placed 4 nodes between $x=0$ and $x=12$.

$$
\begin{array}{cccc}
i = 1 & i = 2 & i = 3 & i = 4 \\
\bullet & \bullet & \bullet & \bullet \\
x = 0 & x = 4 & x = 8 & x = 12
\end{array}
$$
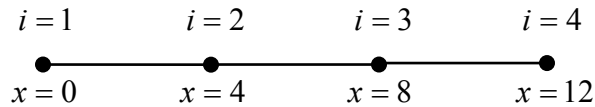
(A)
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0.0625 & 0.125 & 0.0625 & 0 \\
0 & 0.0625 & 0.125 & 0.0625 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}
$$

(B)
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0.0625 & -0.125 & 0.0625 & 0 \\
0 & 0.0625 & -0.125 & 0.0625 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}
$$

(C)
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0.0625 & -0.125 & 0.0625 & 0 \\
0 & 0.0625 & -0.125 & 0.0625
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 16.0 \\ 16.0 \end{bmatrix}
$$

(D)
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0.0625 & 0.125 & 0.0625 & 0 \\
0 & 0.0625 & 0.125 & 0.0625
\end{bmatrix}
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 16.0 \\ 16.0 \end{bmatrix}
$$

Solution
The correct answer is (B).

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2, \ y(0) = 0, \ y(12) = 0, \ 0 \le x \le 12 \qquad (1)$$

The first equation is from the boundary condition
$$y(0) = 0$$

and that is the value of y at node 1, giving
$$y_1 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a)$$

The approximation for $\dfrac{d^2y}{dx^2}$ with a second order accurate central divided difference method is

$$\frac{d^2y}{dx^2} \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

Hence the ordinary differential equation (1) can be approximated at node i as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \approx 6x_i - 0.5x_i^2 \qquad\qquad\qquad\qquad (2)$$

where
$$x_{i+1} = x_i + h$$

At node
$$i = 2,$$
$$x_2 = x_1 + h$$
$$= 0 + 4$$
$$= 4$$

Then at node $i=2$, equation (2) can be written as

$$\frac{y_1 - 2y_2 + y_3}{4^2} = 6 \times 4 - 0.5 \times 4^2$$

$$0.0625y_1 - 0.125y_2 + 0.0625y_3 = 16$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (b)$$

At node
$$i = 3,$$
$$x_3 = x_2 + h$$
$$= 4 + 4$$
$$= 8$$

Then at node $i=3$, equation (2) can be written as

$$\frac{y_2 - 2y_3 + y_4}{4^2} = 6 \times 8 - 0.5 \times 8^2$$

$$0.0625y_2 - 0.125y_3 + 0.0625y_4 = 16$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (c)$$

The final equation is from the boundary condition
$$y(12) = 0$$
or
$$y_4 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (d)$$

Writing equations (a), (b), (c) and (d) in matrix form, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.0625 & -0.125 & 0.0625 & 0 \\ 0 & 0.0625 & -0.125 & 0.0625 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 16.0 \\ 16.0 \\ 0 \end{bmatrix}$$