Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Gaussian Elimination
Simultaneous Linear Equations
COMPLETE SOLUTION SET

1. The goal of forward elimination steps in the Naïve Gauss elimination method is to reduce the coefficient matrix to a (an) _____________ matrix.

   (A) diagonal
   (B) identity
   (C) lower triangular
   (D) upper triangular

Solution
The correct answer is (D).

By reducing the coefficient matrix to an upper triangular matrix, starting from the last equation, each equation can be reduced to one equation-one unknown to be solved by back substitution.

2. Division by zero during forward elimination steps in Naïve Gaussian elimination of the set of equations $[A][X]=[C]$ implies the coefficient matrix $[A]$

(A) is invertible
(B) is nonsingular
(C) may be singular or nonsingular
(D) is singular

Solution
The correct answer is (C).

Division by zero during forward elimination does not relate to whether or not the coefficient matrix is singular or nonsingular. For example

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 12 \\ 12 \end{bmatrix}$$

would give a division by zero error in the first step of forward elimination. However, the coefficient matrix in this case is nonsingular.
In another example

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 7 \\ 4 & 6 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 5 \\ 12 \\ 12 \end{bmatrix}$$

would also give a division by zero error in the first step of forward elimination. In this case the coefficient matrix is singular.

3. Using a computer with four significant digits with chopping, the Naïve Gauss elimination solution to

$$0.0030x_1 + 55.23x_2 = 58.12$$
$$6.239x_1 - 7.123x_2 = 47.23$$

is

(A) $x_1 = 26.66;\ x_2 = 1.051$
(B) $x_1 = 8.769;\ x_2 = 1.051$
(C) $x_1 = 8.800;\ x_2 = 1.000$
(D) $x_1 = 8.771;\ x_2 = 1.052$

Solution

The correct answer is (A).

Writing all the entries with 4 significant digits

$$\begin{bmatrix} 0.003000 & 55.23 \\ 6.239 & -7.123 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 58.12 \\ 47.23 \end{bmatrix}$$

Forward Elimination: Divide *Row* 1 by 0.003000 and multiply it by 6.239, giving the multiplier as $\frac{6.239}{0.003000} = 2079$.

$[Row\ 1] \times 2079$ gives *Row* 1 as

$$[0.003000 \quad 55.23 \quad | \quad 58.12]$$
$$[6.239 \quad 1.148\times 10^5 \quad | \quad 1.208\times 10^5]$$

Subtract the above result from *Row* 2 changes *Row* 2 to

$$[0 \quad -1.148\times 10^5 \quad | \quad -1.207\times 10^5]$$

and hence giving the set of equations at the end of the 1st step of forward elimination as

$$\begin{bmatrix} 0.0030 & 55.23 \\ 0 & -1.148\times 10^5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 58.12 \\ -1.207\times 10^5 \end{bmatrix}$$

This is also the end of all steps of forward elimination.

Back substitution: From the second equation

$$\left(-1.148\times 10^5\right)\times x_2 = -1.207\times 10^5$$

$$x_2 = \frac{-1.207\times 10^5}{-1.148\times 10^5}$$
$$= 1.051$$

From the first equation,

$$0.003000x_1 + 55.23x_2 = 58.12$$

$$
\begin{aligned}
x_1 &= \frac{58.12 - 55.23x_2}{0.003000} \\
&= \frac{58.12 - 55.23(1.051)}{0.0030} \\
&= \frac{58.12 - 58.04}{0.003000} \\
&= \frac{0.08000}{0.003000} \\
&= 26.66
\end{aligned}
$$

4. Using a computer with four significant digits with chopping, the Gaussian elimination with partial pivoting solution to

$$0.0030x_1 + 55.23x_2 = 58.12$$

$$6.239x_1 - 7.123x_2 = 47.23$$

is

(A) $x_1 = 26.66;\ x_2 = 1.051$
(B) $x_1 = 8.769;\ x_2 = 1.051$
(C) $x_1 = 8.800;\ x_2 = 1.000$
(D) $x_1 = 8.771;\ x_2 = 1.052$

Solution

The correct answer is (B).

Writing all the entries with 4 significant digits

$$\begin{bmatrix} 0.003000 & 55.23 \\ 6.239 & -7.123 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 58.12 \\ 47.23 \end{bmatrix}$$

<u>Forward elimination:</u>

Now for the first step of forward elimination, the absolute value of first column elements is

$$|0.003000|, |6.239|$$

or

$$0.003000,\ 6.239$$

So we need to switch *Row* 1 with *Row* 2, to get

$$\begin{bmatrix} 6.239 & -7.123 \\ 0.003000 & 55.23 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 47.12 \\ 58.12 \end{bmatrix}$$

Divide *Row* 1 by 6.239 and multiply it by 0.00300, gives the multiplier as

$$\frac{0.003000}{6.239} = 4.808 \times 10^{-4}$$

$$[Row\ 1] \times\ 4.808 \times 10^{-4}$$

gives *Row* 1 as

$$[2.999 \times 10^{-3} \quad 3.424 \times 10^{-3} \ \mid \quad 2.265 \times 10^{-2}]$$

Subtract the above result from *Row* 2 changes *Row* 2 to

$$[0 \qquad 55.22 \ \mid \quad 58.09]$$

and hence giving the set of equations at the end of the 1st step of forward elimination as

$$\begin{bmatrix} 6.239 & -7.123 \\ 0 & 55.22 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 47.23 \\ 58.09 \end{bmatrix}$$

<u>Back substitution</u>: From the second equation

$$55.22x_2 = 58.09$$

$$x_2 = \frac{58.09}{55.22}$$

$$= 1.051$$

Substituting the value of x_2 in the first equation

$$6.239x_1 - 7.123x_2 = 47.23$$

$$x_1 = \frac{47.23 + 7.123x_2}{6.239}$$

$$= \frac{47.23 + 7.123(1.051)}{6.239}$$

$$= \frac{47.23 + 7.486}{6.239}$$

$$= \frac{54.71}{6.239}$$

$$= 8.769$$

5. At the end of the forward elimination steps of the Naïve Gauss elimination method on the following equations

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 4.2857\times10^{7} & -5.4619\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ -6.5 & -0.15384 & 6.5 & 0.15384 \\ 0 & 0 & 4.2857\times10^{7} & -3.6057\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 0 \\ 0.007 \\ 0 \end{bmatrix}$$

the resulting equations in matrix form are given by

$$\begin{bmatrix} 4.2857\times10^{7} & -9.2307\times10^{5} & 0 & 0 \\ 0 & 3.7688\times10^{5} & -4.2857\times10^{7} & 5.4619\times10^{5} \\ 0 & 0 & -26.9140 & 0.579684 \\ 0 & 0 & 0 & 5.62500\times10^{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} -7.887\times10^{3} \\ 7.887\times10^{3} \\ 1.19530\times10^{-2} \\ 1.90336\times10^{4} \end{bmatrix}$$

The determinant of the original coefficient matrix is

(A) 0.00

(B) 4.2857×10^{7}

(C) 5.486×10^{19}

(D) -2.445×10^{20}

Solution

The correct answer is (D).

If a matrix is upper triangular, lower triangular or diagonal, then the determinant is

$$a_{11}\times a_{22}\times ....\times a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Thus, the determinant, D, of the matrix is

$$D = \left(4.2857\times10^{7}\right)\times\left(3.7688\times10^{5}\right)\times\left(-26.9140\right)\times\left(5.62500\times10^{5}\right)$$

$$= -2.445\times10^{20}$$

6. The following data is given for the velocity of the rocket as a function of time. To find the velocity at $t = 21\,\text{s}$, you are asked to use a quadratic polynomial, $v(t) = at^2 + bt + c$ to approximate the velocity profile.

t	(s)	0	14	15	20	30	35
$v(t)$	(m/s)	0	227.04	362.78	517.35	602.97	901.67

The correct set of equations that will find a, b and c are

(A) $$\begin{bmatrix} 176 & 14 & 1 \\ 225 & 15 & 1 \\ 400 & 20 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix}$$

(B) $$\begin{bmatrix} 225 & 15 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$

(C) $$\begin{bmatrix} 0 & 0 & 1 \\ 225 & 15 & 1 \\ 400 & 20 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 362.78 \\ 517.35 \end{bmatrix}$$

(D) $$\begin{bmatrix} 400 & 20 & 1 \\ 900 & 30 & 1 \\ 1225 & 35 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 517.35 \\ 602.97 \\ 901.67 \end{bmatrix}$$

Solution

The correct answer is (B).

First choose the three points closest to $t = 21\,\text{s}$ that also bracket it.

$$t_0 = 15\,\text{s}, v(t_0) = 362.78\,\text{m/s}$$
$$t_1 = 20\,\text{s}, v(t_1) = 517.35\,\text{m/s}$$
$$t_2 = 30\,\text{s}, v(t_2) = 602.97\,\text{m/s}$$

Such that

$$v(15) = 362.78 = a(15)^2 + b(15) + c$$
$$v(20) = 517.35 = a(20)^2 + b(20) + c$$
$$v(30) = 602.97 = a(30)^2 + b(30) + c$$

This expands to

$$225a + 15b + c = 362.78$$
$$400a + 20b + c = 517.35$$
$$900a + 30b + c = 602.97$$

$$\begin{bmatrix} 225a + 15b + c \\ 400a + 20b + c \\ 900a + 30b + c \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$

This can be rewritten as

$$\begin{bmatrix} 225 & 15 & 1 \\ 400 & 20 & 1 \\ 900 & 30 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$