Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Chapter 04.09
Adequacy of Solutions
COMPLETE SOLUTION SET

1. The row sum norm of the matrix

$$[A] = \begin{bmatrix} 6 & -7 & 3 & 13 \\ 19 & -21 & 23 & -29 \\ 41 & 47 & -51 & 61 \end{bmatrix}$$

is
(A) 29
(B) 61
(C) 98
(D) 200

Solution
The correct answer is (*D*).

The row sum norm of a matrix $[A]$ of size $m \times n$ is given by

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

For the example above, $[A]$ is a matrix of order 3×4, hence $m = 3$, $n = 4$.

For Row 1

$$\sum_{j=1}^{4} |a_{1j}| = |6| + |-7| + |3| + |13|$$
$$= 29$$

For Row 2

$$\sum_{j=1}^{4} |a_{2j}| = |19| + |-21| + |23| + |-29|$$
$$= 92$$

For Row 3

$$\sum_{j=1}^{4} |a_{3j}| = |41| + |47| + |-51| + |61|$$
$$= 200$$

The maximum of $(29, 92, 200) = 200$. Hence

$$\|A\|_{\infty} = 200$$

2. The adequacy of the solution of simultaneous linear equations $[A][X]=[C]$ depends on
   (A) the condition number of coefficient matrix $[A]$
   (B) the machine epsilon
   (C) the condition number for matrix $[A]$ and the machine epsilon
   (D) norm of the coefficient matrix $[A]$

Solution

The correct answer is (C).

The adequacy of the solution of the coefficient matrix depends on the condition number as well as the machine epsilon of the computing machine.

For a set of equations given by $[A][X]=[C]$, the relative change in the norm of the solution vector is related to the relative change in the norm of the right hand side vector by

$$\frac{\|\Delta X\|}{\|X\|} \le cond(A)\frac{\|\Delta C\|}{\|C\|}.$$

The condition number of the coefficient matrix is the upper limit of the amplification of the relative change in the norm of the solution vector, and hence is the measure of the adequacy of the solution. The machine epsilon is a measure of the relative error in the round-off error caused by approximate representation of numbers in a computing device.

Relative size of error in solution of a set of equations is proportional to the product $cond(A)\times \varepsilon_{mach}$

3. Given a set of equations in matrix form $[A][X]=[C]$, $\|A\|=250, \|A^{-1}\|=40$ and $\varepsilon_{mach}=0.119\times10^{-6}$, then the number of significant digits you can at least trust in the solution are
   (A) 1
   (B) 2
   (C) 3
   (D) 4

Solution

The correct answer is (*B*).

$$cond(A)=\|A\|\|A^{-1}\|$$
$$=250\times40$$
$$=10000$$

Relative size of error in solution

$$\propto cond(A)\times\varepsilon_{mach}$$
$$=(10000)\times(0.119\times10^{-6})$$
$$=0.119\times10^{-2}$$

Comparing

$$0.119\times10^{-2}\leq0.5\times10^{-m}$$
$$m\geq2$$

we can expect the solution to be correct to at least 2 significant digits.

4. The solution to a set of simultaneous linear equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 94 \\ 138 \end{bmatrix}$$

is given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

The solution to another set of simultaneous linear equations is given by (note the coefficient matrix is the same as above)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 43.99 \\ 93.98 \\ 138.03 \end{bmatrix}$$

is given as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 214.01 \\ -208.01 \\ 60 \end{bmatrix}$$

Based on the row sum norm, the condition number of the coefficient matrix is greater than (choose the largest possible value)

(A) 1
(B) 138
(C) 4500
(D) 139320

Solution

The correct answer is (*D*).

$$\frac{\|\Delta X\|_\infty}{\|X\|_\infty} \le cond(A) \frac{\|\Delta C\|_\infty}{\|C\|_\infty}$$

For

$$[C] = \begin{bmatrix} 44 \\ 94 \\ 138 \end{bmatrix}$$

$$\|C\|_\infty = \max(|44|,|94|,|138|) = 138$$

For

$$[X] = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$\|X\|_\infty = \max(|\,2\,|,|\,4\,|,|\,7\,|)$$
$$= 7$$

For

$$[\Delta X] = \begin{bmatrix} 214.01 \\ -208.01 \\ 60 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$
$$= \begin{bmatrix} 212.01 \\ -212.01 \\ 53 \end{bmatrix}$$

$$\|\Delta X\|_\infty = \max(|212.01|,|-212.01|,|53|)$$
$$= 212.01$$

For

$$[\Delta C] = \begin{bmatrix} 43.99 \\ 93.98 \\ 138.03 \end{bmatrix} - \begin{bmatrix} 44 \\ 94 \\ 138 \end{bmatrix}$$
$$= \begin{bmatrix} -0.01 \\ -0.02 \\ 0.03 \end{bmatrix}$$

$$\|\Delta C\|_\infty = \max(|-0.01|,|-0.02|,|0.03|)$$
$$= 0.03$$

$$cond(A) \geq \frac{\|\Delta X\|_\infty / \|X\|_\infty}{\|\Delta C\|_\infty / \|C\|_\infty}$$
$$= \frac{212.01/7}{0.03/138}$$
$$= 139320$$

5. The condition number of the $n \times n$ identity matrix based on the row sum norm is
   (A) 0
   (B) 1
   (C) n
   (D) n^2

Solution

The correct answer is (B).

$$cond(A) = \|A\| \|A^{-1}\|$$

If A is an identity matrix, the row sum norm of $[A]$ is

$$\|A\|_\infty = 1$$

If A is an identity matrix, $[A]^{-1} = [I]$, the row sum norm of $[A]^{-1}$ is

$$\|A^{-1}\|_\infty = 1$$

$$\begin{aligned} cond(A) &= \|A\| \|A^{-1}\| \\ &= (1)(1) \\ &= 1 \end{aligned}$$

6. Let $[A]=\begin{bmatrix} 1 & 2+\delta \\ 2-\delta & 1 \end{bmatrix}$. Based on the row sum norm and given that $\delta \to 0, \delta > 0$, the condition number of the matrix is

(A) $\dfrac{3-\delta}{3+\delta}$

(B) $\dfrac{9-\delta^2}{3-\delta^2}$

(C) $\dfrac{(3+\delta)^2}{3-\delta^2}$

(D) $\dfrac{3-2\delta-\delta^2}{3-\delta^2}$

Solution

The correct answer is (C).

The row sum norm of a matrix $[A]$ by size $m \times n$ is given by

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

For Row 1

$$\sum_{j=1}^{2} |a_{1j}| = |1| + |2+\delta|$$
$$= 3+\delta$$

For Row 2

$$\sum_{j=1}^{2} |a_{1j}| = |2-\delta| + |1|$$
$$= 3-\delta$$

The row sum norm of $[A]$ is

$$\|A\|_\infty = \max(3+\delta, 3-\delta)$$
$$= 3+\delta$$

The inverse of $[A]$ is

$$[A]^{-1} = \begin{bmatrix} \dfrac{1}{-3+\delta^2} & -\dfrac{2+\delta}{-3+\delta^2} \\ \dfrac{-2+\delta}{-3+\delta^2} & \dfrac{1}{-3+\delta^2} \end{bmatrix}$$

For Row 1

$$\sum_{j=1}^{2}\left|a_{1j}^*\right| = \left|\frac{1}{-3+\delta^2}\right| + \left|-\frac{2+\delta}{-3+\delta^2}\right|$$

$$= \frac{1}{3-\delta^2} + \frac{2+\delta}{3-\delta^2}$$

$$= \frac{3+\delta}{3-\delta^2}$$

For Row 2

$$\sum_{j=1}^{2}\left|a_{2j}^*\right| = \left|\frac{-2+\delta}{-3+\delta^2}\right| + \left|\frac{1}{-3+\delta^2}\right|$$

$$= \frac{2-\delta}{3-\delta^2} + \frac{1}{3-\delta^2}$$

$$= \frac{3-\delta}{3-\delta^2}$$

$$\left\|A^{-1}\right\|_\infty = \max\left(\frac{3+\delta}{3-\delta^2}, \frac{3-\delta}{3-\delta^2}\right)$$

$$= \frac{3+\delta}{3-\delta^2}$$

$$cond(A) = \|A\|\left\|A^{-1}\right\|$$

$$= (3+\delta)\frac{3+\delta}{3-\delta^2}$$

$$= \frac{(3+\delta)^2}{3-\delta^2}$$