System of equations

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Objectives

After reading this chapter, you should be able to:

1. *setup simultaneous linear equations in matrix form and vice-versa,*
2. *understand the concept of the inverse of a matrix,*
3. *know the difference between a consistent and inconsistent system of linear equations, and*
4. *learn that a system of linear equations can have a unique solution, no solution or infinite solutions.*

Systems of Equations

Matrix algebra is used for solving systems of equations. Can you illustrate this concept?

Matrix algebra is used to solve a system of simultaneous linear equations. In fact, for many mathematical procedures such as the solution to a set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations. Let us illustrate with an example for interpolation.

Example 1

The upward velocity of a rocket is given at three different times on the following table.

Table 5.1. Velocity vs. time data for a rocket

Time, *t*	Velocity, *v*
(s)	(m/s)
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as

$$v(t) = at^2 + bt + c, \quad 5 \le t \le 12.$$

Set up the equations in matrix form to find the coefficients a, b, c of the velocity profile.

Example 1 (cont.)

The polynomial is going through three data (t_1, v_1), (t_2, v_2), and (t_3, v_3) where from table 5.1.

$$t_1 = 5, v_1 = 106.8$$

$$t_2 = 8, v_2 = 177.2$$

$$t_3 = 12, v_3 = 279.2$$

Requiring that $v(t) = at^2 + bt + c$ passes through the three data points gives

$$v(t_1) = v_1 = at_1^2 + bt_1 + c$$

$$v(t_2) = v_2 = at_2^2 + bt_2 + c$$

$$v(t_3) = v_3 = at_3^2 + bt_3 + c$$

Example 1 (cont.)

Substituting the data $(t_1, v_1), (t_2, v_2),$ and (t_3, v_3) gives

$$a(5^2) + b(5) + c = 106.8$$

$$a(8^2) + b(8) + c = 177.2$$

$$a(12^2) + b(12) + c = 279.2$$

or

$$25a + 5b + c = 106.8$$

$$64a + 8b + c = 177.2$$

$$144a + 12b + c = 279.2$$

Example 1 (cont.)

This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25a + & 5b + & c \\ 64a + & 8b + & c \\ 144a + & 12b + & c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

The above equation can be written as a linear combination as follows

$$a\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + b\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and further using matrix multiplication gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 (cont.)

The previous is an illustration of why matrix algebra is needed. The complete solution to the set of equations is given later in this chapter.

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

Example 1 (cont.)

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}$$

Denoting the matrices by $[A]$, $[X]$, and $[C]$, the system of equation is

$[A][X]=[C]$, where $[A]$ is called the coefficient matrix, $[C]$ is called the right hand side vector and $[X]$ is called the solution vector.

Example 1 (cont.)

Sometimes $[A][X]=[C]$ systems of equations are written in the augmented form. That is

$$[A \vdots C] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & \vdots c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & \vdots c_2 \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & \vdots c_n \end{bmatrix}$$

Consistent and inconsistent system

A system of equations can be consistent or inconsistent. What does that mean?

A system of equations $[A][X]=[C]$ is consistent if there is a solution, and it is inconsistent if there is no solution. However, a consistent system of equations does not mean a unique solution, that is, a consistent system of equations may have a unique solution or infinite solutions (Figure 1).

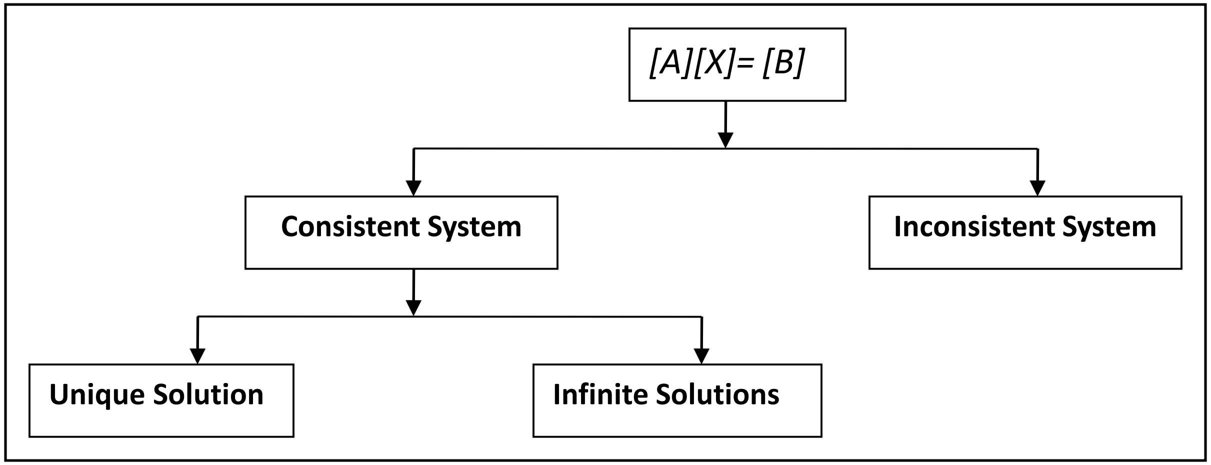

Figure 5.1. Consistent and inconsistent system of equations flow chart.

Example 2

Give examples of consistent and inconsistent system of equations.

Solution

a) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is a consistent system of equations as it has a unique solution, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example 2 (cont.)

b) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

is also a consistent system of equations but it has infinite solutions as given as follows. Expanding the above set of equations,

$$2x + 4y = 6$$

$$x + 2y = 3$$

Example 2 (cont.)

you can see that they are the same equation. Hence, any combination of (x, y) that satisfies,

$$2x + 4y = 6$$

is a solution. For example $(x, y) = (1,1)$ is a solution. Other solutions include

$(x, y) = (0.5, 1.25)$, $(x, y) = (0,\ 1.5)$, and so on.

c) The system of equations

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

is inconsistent as no solutions exists.

Distinguishing consistency

How can one distinguish between a consistent and inconsistent system of equations?

A system of equations $[A][X]=[C]$ is *consistent* if the rank of A is equal to the rank of the augmented matrix $[A \vdots C]$.

A system of equations $[A][X]=[C]$ is *inconsistent* if the rank of A is less than the rank of the augmented matrix $[A \vdots C]$.

But, what do you mean by rank of a matrix?

The rank of a matrix is defined as the order of the largest square submatrix whose determinant is not zero.

Example 3

What is the rank of

$$[A] = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix} \quad ?$$

Solution

The largest square submatrix possible is of order 3 and that is $[A]$ itself. Since $\det(A) = -23 \neq 0,$ the rank of $[A] = 3.$

Example 4

What is the rank of

$$[A]=\begin{bmatrix}3 & 1 & 2\\ 2 & 0 & 5\\ 5 & 1 & 7\end{bmatrix}?$$

Solution

The largest square submatrix of $[A]$ is of order 3 and that is $[A]$ itself. Since $\det(A)=0$, the rank of $[A]$ is less than 3. The next largest square submatrix would be a 2×2 matrix. One of the square submatrices of $[A]$ is

$$[B]=\begin{bmatrix}3 & 1\\ 2 & 0\end{bmatrix}$$

And $\det(B)=-2\neq 0$. Hence the rank of $[A]$ is 2. There is no need to look at other 2×2 submatrices to establish that the rank of $[A]$ is 2.

Example 5

How do I now use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent or inconsistent system of equations?

Example 5 (cont.)

Solution

The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side vector is

Example 5 (cont.)

The augmented matrix is

$$[B]=\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Since there are no square submatrices of order 4 as $[B]$ is a 3×4 matrix, the rank of $[B]$ is at most 3. So let us look at the square submatrices of $[B]$ of order 3; if any of these square submatrices have determinant not equal to zero, then the rank is 3. For example, a submatrix of the augmented matrix $[B]$ is

Example 5 (cont.)

$$[D]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

has. $\det(D)=-84\neq 0$
Hence the rank of the augmented matrix $[B]$ is 3. Since $[A]=[D]$, the rank of $[A]$ is 3.
Since the rank of the augmented matrix $[B]$ equals the rank of the coefficient matrix $[A]$, the system of equations is consistent.

Example 6

Use the concept of rank of matrix to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

Is consistent or inconsistent?

Example 6 (cont.)

Solution

The coefficient matrix is given by

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

and the right hand side

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

Example 6 (cont.)

Since there are no square submatrices of order 4 as $[B]$ is a 4×3 matrix, the rank of the augmented $[B]$ is at most 3. So let us look at square submatrices of the augmented matrix $[B]$ of order 3 and see if any of these have determinants not equal to zero. For example, a square submatrix of the augmented matrix $[B]$ is

$$[D]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix}$$

Has $\det(D)=0$. This means, we need to explore other square submatrices of order 3 of the augmented matrix $[B]$ and find their determinants.

Example 6 (cont.)

That is,

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 284.0 \end{bmatrix}$$

$$\det(E) = 0$$

$$[F] = \begin{bmatrix} 25 & 5 & 106.8 \\ 64 & 8 & 177.2 \\ 89 & 13 & 284.0 \end{bmatrix}$$

$$\det(F) = 0$$

$$[G] = \begin{bmatrix} 25 & 1 & 106.8 \\ 64 & 1 & 177.2 \\ 89 & 2 & 284.0 \end{bmatrix}$$

$$\det(G) = 0$$

Example 6 (cont.)

All the square submatrices of order 3×3 of the augmented matrix $[B]$ have a zero determinant. So the rank of the augmented matrix $[B]$ is less than 3. Is the rank of augmented matrix $[B]$ equal to 2? One of the 2×2 submatrices of the augmented matrix $[B]$ is

$$[H]=\begin{bmatrix}25 & 5\\ 64 & 8\end{bmatrix}$$

And

$$\det(H)=-120\neq 0$$

So the rank of the augmented matrix $[B]$ is 2.

Example 6 (cont.)

Now we need to find the rank of the coefficient matrix $[B]$

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$$

and

$$\det(A)=0$$

So the rank of the coefficient matrix $[A]$ is less than 3. A square submatrix of the coefficient matrix $[A]$ is

$$[J]=\begin{bmatrix}5 & 1\\ 8 & 1\end{bmatrix}$$

$$\det(J)=-3\neq 0$$

So the rank of the coefficient matrix $[A]$ is 2.

Example 6 (cont.)

Hence, the rank of the coefficient matrix $[A]$ equals the rank of the augmented matrix $[B]$. So the system of equations $[A][X]=[C]$ is consistent.

Example 7

Use the concept of rank to find if

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 280.0 \end{bmatrix}$$

is consistent or inconsistent

Example 7 (cont.)

The augmented matrix is

$$[B]=\begin{bmatrix}25 & 5 & 1 & :106.8\\ 64 & 8 & 1 & :177.2\\ 89 & 13 & 2 & :280.0\end{bmatrix}$$

Since there are no square submatrices of order 4×4 as the augmented matrix $[B]$ is a 4×3 matrix, the rank of the augmented matrix $[B]$ is at most 3. So let us look at square submatrices of the augmented matrix (B) of order 3 and see if any of the 3×3 submatrices have a determinant not equal to zero. For example, a square submatrix of order 3×3 of $[B]$

$$[D]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 89 & 13 & 2\end{bmatrix}$$

$$\det(D) = 0$$

Example 7 (cont.)

So it means, we need to explore other square submatrices of the augmented matrix $[B]$

$$[E] = \begin{bmatrix} 5 & 1 & 106.8 \\ 8 & 1 & 177.2 \\ 13 & 2 & 280.0 \end{bmatrix}$$

$$\det(E0 \neq 12.0 \neq 0$$

So the rank of the augmented matrix $[B]$ is 3.
The rank of the coefficient matrix $[A]$ is 2 from the previous example.
Since the rank of the coefficient matrix $[A]$ is less than the rank of the augmented matrix $[B]$, the system of equations is inconsistent. Hence, no solution exists for $[A][X] = [C]$

If a solution exists, how do we know whether it is unique?

In a system of equations $[A][X]=[C]$ that is consistent, the rank of the coefficient matrix $[A]$ is the same as the augmented matrix $[A|C]$. If in addition, the rank of the coefficient matrix $[A]$ is same as the number of unknowns, then the solution is unique; if the rank of the coefficient matrix $[A]$ is less than the number of unknowns, then infinite solutions exist.

Flowchart of conditions

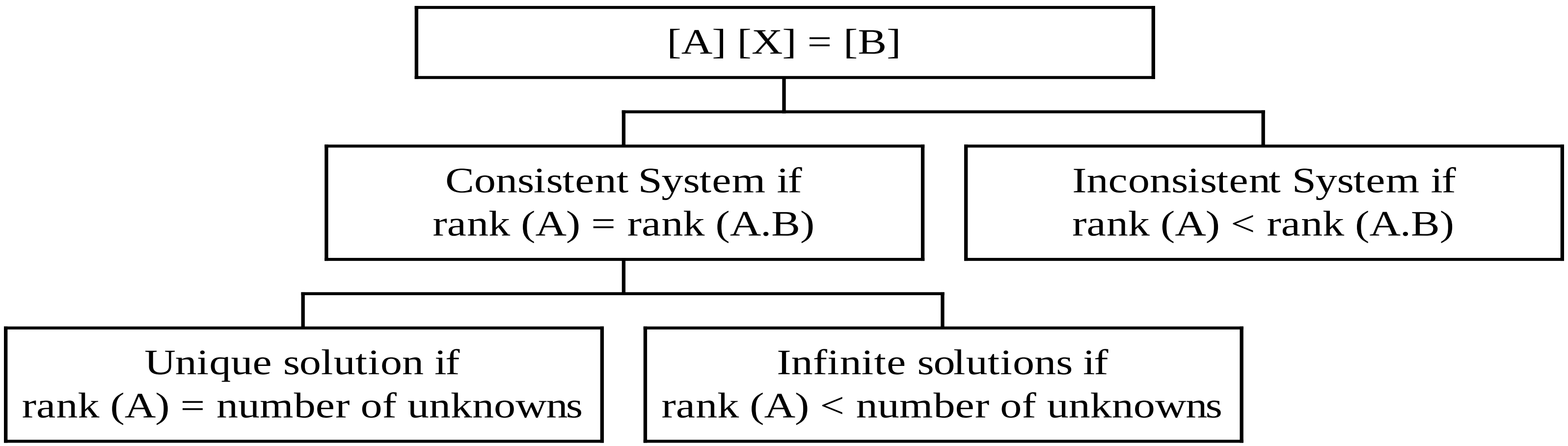

Figure 5.2. Flow chart of conditions for consistent and inconsistent system of equations.

Example 8

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

is a consistent system of equations. Does the system of equations have a unique solution or does it have infinite solutions?

Example 8 (cont.)

Solution

The coefficient matrix is

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and the right hand side is

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

While finding out whether the above equations were consistent in an earlier example, we found that the rank of the coefficient matrix (A) equals rank of augmented matrix $[A \vdots C]$ equals 3.

The solution is unique as the number of unknowns = 3 = rank of (A).

Example 9

We found that the following system of equations

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 284.0 \end{bmatrix}$$

is a consistent system of equations. Is the solution unique or does it have infinite solutions.

Example 9 (cont.)

Solution

While finding out whether the above equations were consistent, we found that the rank of the coefficient matrix $[A]$ equals the rank of augmented matrix $(A \vdots C)$ equals 2

Since the rank of $[A] = 2$ < number of unknowns = 3, infinite solutions exist.

If we have more equations than unknowns in [A] [X] = [C], does it mean the system is inconsistent?

No, it depends on the rank of the augmented matrix $[A \vdots C]$ and the rank of $[A]$

Example 9 (cont.)

a) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 284.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 3

rank of coefficient matrix = 3

Now since the rank of (A) = 3 = number of unknowns, the solution is not only consistent but also unique.

Example 9 (cont.)

b) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \\ 280.0 \end{bmatrix}$$

is inconsistent, since

rank of augmented matrix = 4

rank of coefficient matrix = 3

Example 9 (cont.)

c) For example

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 50 & 10 & 2 \\ 89 & 13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 213.6 \\ 280.0 \end{bmatrix}$$

is consistent, since

rank of augmented matrix = 2

rank of coefficient matrix = 2

But since the rank of (A) = 2 < the number of unknowns = 3, infinite solutions exist.

Example 9 (cont.)

Consistent systems of equations can only have a unique solution or infinite solutions. Can a system of equations have more than one but not infinite number of solutions?

No, you can only have either a unique solution or infinite solutions. Let us suppose

$[A][X] = [C]$ has two solutions $[Y]$ and $[Z]$ so that

$$[A][Y] = [C]$$

$$[A][Z] = [C]$$

Example 9 (cont.)

If r is a constant, then from the two equations

$$r[A][Y] = r[C]$$

$$(1-r)[A][Z] = (1-r)[C]$$

Adding the above two equations gives

$$r[A][Y] + (1-r)[A][Z] = r[C] + (1-r)[C]$$

$$[A](r[Y] + (1-r)[Z]) = [C]$$

Example 9 (cont.)

Hence

$$r[Y]+(1-r)[Z]$$

is a solution to

$$[A][X]=[C]$$

Since r is any scalar, there are infinite solutions for $[A][X]=[C]$ of the form

$$r[Y]+(1-r)[Z]$$

Can you divide two matrices?

If $[A][B]=[C]$ is defined, it might seem intuitive that $[A]=\frac{[C]}{[B]}$,

but matrix division is not defined like that. However an inverse of a matrix can be defined for certain types of square matrices. The inverse of a square matrix $[A]$, if existing, is denoted by $[A]^{-1}$ such that

$$[A][A]^{-1}=[I]=[A]^{-1}[A]$$

Where $[I]$ is the identity matrix.

Can you divide two matrices?

In other words, let [A] be a square matrix. If [B] is another square matrix of the same size such that $[B][A]=[I]$, then [B] is the inverse of [A]. [A] is then called to be invertible or nonsingular. If $[A]^{-1}$ does not exist, [A] is called noninvertible or singular. If [A] and [B] are two $n\times n$ matrices such that $[B][A]=[I]$, then these statements are also true

- [B] is the inverse of [A]
- [A] is the inverse of [B]
- [A] and [B] are both invertible
- [A] [B]=[I].
- [A] and [B] are both nonsingular
- all columns of [A] and [B]are linearly independent
- all rows of [A] and [B] are linearly independent.

Example 10

Determine if

$$[B] = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

is the inverse of

$$[\mathrm{A}] = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Example 10 (cont.)

Solution

$$[B][A] = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= [I]$$

Since,

$$[B][A] = [I]$$

$[B]$ is the inverse of $[A]$ and $[A]$ is the inverse of $[B]$.

Using the inverse of a matrix

Can I use the concept of the inverse of a matrix to find the solution of a set of equations [A] [X] = [C]?

Yes, if the number of equations is the same as the number of unknowns, the coefficient matrix $[A]$ is a square matrix.
Given

$$[A][X]=[C]$$

Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$.

$$[A]^{-1}[A][X]=[A]^{-1}[C]$$

$$[I][X]=[A]^{-1}[C]$$

$$[X]=[A]^{-1}[C]$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of $[A][X]=[C]$ is simply a multiplication of $[A]^{-1}$ and the right hand side vector, $[C]$

How do I find the inverse of a matrix?

If $[A]$ is a $n \times n$ matrix, then $[A]^{-1}$ is a $n \times n$ matrix and according to the definition of inverse of a matrix

$$[A][A]^{-1} = [I]$$

Denoting

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

How do I find the inverse of a matrix? (cont.)

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & a'_{nn} \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & & & & 0 \\ 0 & & \cdot & & & \cdot \\ \cdot & & & 1 & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

How do I find the inverse of a matrix? (cont.)

Using the definition of matrix multiplication, the first column of the $[A]^{-1}$ matrix can then be found by solving

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ \cdot \\ \cdot \\ a'_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Similarly, one can find the other columns of the $[A]^{-1}$ matrix by changing the right hand side accordingly.

Example 11

The upward velocity of the rocket is given by

Table 5.2. Velocity vs time data for a rocket

Time, t (s)	Velocity, v (m/s)
5	106.8
8	177.2
12	279.2

In an earlier example, we wanted to approximate the velocity profile by

$$v(t) = at^2 + bt + c, \quad 5 \le t \le 12$$

Example 11 (cont.)

We found that the coefficients $a, b, \text{and}\, c$ in $v(t)$ are given by

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

First, find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and then use the definition of inverse to find the coefficients $a, b, \text{and}\, c$

Example 11 (cont.)

Solution

If

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$$

is the inverse of $[A]$, then

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 11 (cont.)

Gives three sets of equations,

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 11 (cont.)

Solving the above three sets of equations separately gives

$$\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example 11 (cont.)

Hence

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Now

$$[A][X]=[C]$$

where

$$[X]=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[C]=\begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 11 (cont.)

Using the definition of $[A]^{-1}$,

$$[A]^{-1}[A][X] = [A]^{-1}[C]$$

$$[X] = [A]^{-1}[C]$$

$$\begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix} \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 11 (cont.)

Hence

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2905 \\ 19.69 \\ 1.086 \end{bmatrix}$$

So

$$v(t) = 0.2905t^2 + 19.69t + 1.086,\ 5 \le t \le 12$$

Is there another way to find the inverse of a matrix?

For finding the inverse of small matrices, the inverse of an invertible matrix can be found by

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$

Where

$$adj(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{T}$$

Is there another way to find the inverse of a matrix? (cont.)

where C_{ij} are the cofactors of a_{ij}. The matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & \cdots & \cdots & C_{nn} \end{bmatrix}$$

itself is called the matrix of cofactors from $[A]$. Cofactors are defined in Chapter 4.

Example 12

Find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

From Example 4.6 in Chapter 04.06, we found

$$\det(A) = -84$$

Next we need to find the adjoint of $[A]$. The cofactors of are found as follows.

Example 12 (cont.)

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 \\ 12 & 1 \end{vmatrix}$$

$$= -4$$

The cofactors of entry a_{11} is

$$C_{11} = (-1)^{1+1} M_{11}$$

$$= M_{11}$$

$$= -4$$

Example 12 (cont.)

The minor of entry a_{12} is

$$M_{12} = \begin{vmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 64 & 1 \\ 144 & 1 \end{vmatrix}$$

$$= -80$$

The cofactors of entry a_{12} is

$$C_{12} = (-1)^{1+2} M_{12}$$

$$= -M_{12}$$

$$= -(-80)$$

$$= 80$$

Example 12 (cont.)

Similarly

$$C_{13} = -384$$

$$C_{21} = 7$$

$$C_{22} = -119$$

$$C_{23} = 420$$

$$C_{31} = -3$$

$$C_{32} = 39$$

$$C_{33} = -120$$

Example 12 (cont.)

Hence the matrix of cofactors of $[A]$ is

$$[C]=\begin{bmatrix} -4 & 80 & -384 \\ 7 & -119 & 420 \\ -3 & 39 & -120 \end{bmatrix}$$

The adjoint of matrix $[A]$ is $[C]^{T}$,

$$adj(A)=[C]^{T}$$

$$=\begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

Example 12 (cont.)

Hence

$$[A]^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{-84}\begin{bmatrix} -4 & 7 & -3 \\ 80 & -119 & 39 \\ -384 & 420 & -120 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

If the inverse of a square matrix [A] exists, is it unique?

Yes, the inverse of a square matrix is unique, if it exists. The proof is as follows. Assume that the inverse of $[A]$ is $[B]$ and if this inverse is not unique, then let another inverse of $[A]$ exist called $[C]$

If $[B]$ is the inverse of $[A]$, then

$$[B][A]=[I]$$

Multiply both sides by $[C]$

$$[B][A][C]=[I][C]$$

$$[B][A][C]=[C]$$

If the inverse of a square matrix [A] exists, is it unique? (cont.)

Since $[C]$ is inverse of $[A]$

$$[A][C]=[I]$$

Multiply both sides by $[B]$

$$[B][I]=[C]$$

$$[B]=[C]$$

This shows that $[B]$ and $[C]$ are the same. So the inverse of $[A]$ is unique.

Key terms

Consistent system
Inconsistent system
Infinite solutions
Unique solution
Rank
Inverse